

# Cancellation of divergences in $\mathcal{N} = 4$ SYM/Type IIB Supergravity correspondence.

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## Abstract

Using Schrödinger functional methods, we show that in the  $\mathcal{N} = 4$  SYM/Type IIB Supergravity correspondence the renormalisation of the boundary Newton and gravitational constants arising from bulk fields cancels when we sum over all the Kaluza-Klein modes of Supergravity. This accords with the expected finiteness of  $\mathcal{N} = 4$  SYM, and it is expected that other renormalisations cancel in a similar way.

The correspondence between  $\mathcal{N} = 4$  Super-Yang-Mills Theory and Type IIB Supergravity/String Theory [1] has been of considerable importance in shedding light on both theories. Most of the work that has been done to date has focussed on the large- $N$  limit of the SYM theory, largely because the calculation of string loops on the AdS background is not well understood. However, many important subleading order effects in the large- $N$  expansion correspond to Supergravity loops and can be calculated. An example of this is the Weyl anomaly, which receives contributions at one loop from all of the Kaluza-Klein modes of Supergravity [3, 5].

Loop effects in supergravity also renormalise the boundary wave-functional, which according to the correspondence is identified with the partition function of the boundary theory. But the finiteness of the boundary theory leads us to expect that such renormalisations should disappear when the full theory is taken into account.

The purpose of this letter is to demonstrate this cancellation of divergences for the contributions of bulk supergravity fields to the renormalisation of the boundary Newton and cosmological constants. When we sum over all the Kaluza-Klein modes of Type IIB Supergravity compactified on  $AdS_5 \times S^5$  the contributions to the renormalisation cancel, so that no renormalisation is needed when all the bulk modes are taken into account.

The way in which this cancellation happens is significant. For Ricci-flat boundaries the bulk metric is unaffected by introducing boundary curvature and the cancellation happens within supermultiplets. For non-Ricci flat boundaries the bulk metric acquires an extra factor (though the bulk metric satisfies the same Einstein equations) and the cancellation requires an analytic regularisation of the infinite sum over Kaluza-Klein modes. The latter case demonstrates that it is insufficient to consider only the consistent truncation to the massless multiplet of Type IIB Supergravity. So, for example, for a non-Ricci flat boundary, the calculation of the anomaly [3, 5] to the truncated spectrum of [9] fails to produce the expected subleading correction to the coefficient  $c$  for the infra-red fixed point of the RG flow driven by adding certain mass terms to the  $\mathcal{N} = 4$  Super-Yang-Mills theory to break the supersymmetry down to  $\mathcal{N} = 1$ . For a Ricci-flat boundary, however, the truncated spectrum gives the correct result [6].

We expect that other renormalisations due to bulk interactions cancel in a similar way to the cancellations that we describe here.

The bulk  $AdS_5$  metric giving a general Einstein metric  $\hat{g}$  on the boundary is

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = dr^2 + z^{-2} e^\rho \hat{g}_{ij}(x) dx^i dx^j, \quad e^{\rho/2} = 1 - C z^2, \quad C = \frac{l^2 \hat{R}}{48} \quad z = \exp(r/l), \quad (1)$$

and a regularisation is introduced by putting the boundary at  $z = \tau = \exp(r_0/l)$ . Consider a scalar field of mass  $m$  propagating in this metric; it has the action

$$\begin{aligned} S_\phi &= \frac{1}{2} \int d^5 X \sqrt{G} \left( G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) \\ &= \frac{1}{2} \int \frac{d^4 x dr}{z^4} \sqrt{\hat{g}} e^{2\rho} \left( \dot{\phi}^2 + z^2 e^{-\rho} \hat{g}^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2 \right), \end{aligned} \quad (2)$$

with the dot denoting differentiation with respect to  $r$ . The norm on fluctuations of the field from which the functional integral volume element can be constructed is

$$||\delta\phi||^2 = \int d^5 X \sqrt{G} \delta\phi^2 = \int \frac{d^4 x dr}{z^4} \sqrt{\hat{g}} e^{2\rho} \delta\phi^2, \quad (3)$$

and it is convenient to redefine the field by setting  $\phi = z^2 e^{-\rho} \varphi$  to make the ‘kinetic’ term in the action into the standard form. The action becomes

$$S_\phi = \frac{1}{2} \int d^4 x dr \sqrt{\hat{g}} \left( \dot{\varphi}^2 + z^2 e^{-\rho} \varphi \left( \square + \frac{\hat{R}}{6} \right) \varphi + \left( m^2 + \frac{4}{l^2} \right) \varphi^2 \right), \quad (4)$$

where we discarded a boundary term that is eventually sent to zero by wave-function renormalisation [7]. According to the AdS/CFT correspondence the boundary partition function is

$$\Psi = \int \mathcal{D}\varphi e^{-S_\varphi} \Big|_{\varphi(r=r_0)=\hat{\varphi}} \equiv e^{W[\hat{\varphi}]} \quad W[\hat{\varphi}] = F + \frac{1}{2} \int d^4 x \sqrt{\hat{g}} \hat{\varphi} \Gamma \hat{\varphi}, \quad (5)$$

where  $\Gamma$  is a differential operator and  $F$  is the free energy of the scalar field. This satisfies a functional Schrödinger equation that can be read off from the action

$$\frac{\partial}{\partial r_0} \Psi = -\frac{1}{2} \int d^4 x \sqrt{\hat{g}} \left\{ -\hat{g}^{-1} \frac{\delta^2}{\delta \varphi^2} + \tau^2 e^{-\rho} \varphi \left( \square + \frac{\hat{R}}{6} \right) \varphi + \left( m^2 + \frac{4}{l^2} \right) \varphi^2 \right\} \Psi. \quad (6)$$

which implies that

$$\frac{\partial}{\partial r_0} \Gamma = \Gamma^2 - \tau^2 e^{-\rho} \left( \square + \frac{\hat{R}}{6} \right) - \left( m^2 + \frac{4}{l^2} \right), \quad \frac{\partial}{\partial r_0} F = \frac{1}{2} \text{Tr } \Gamma. \quad (7)$$

We can solve for  $\Gamma$  in powers of the differential operator by expanding

$$\Gamma = \sum_{n=0}^{\infty} b_n(r_0) \left( \square + \hat{R}/6 \right)^n, \quad (8)$$

so that

$$b_0 = -\sqrt{m^2 + \frac{4}{l^2}}, \quad (9)$$

(we take the minus sign to give a normalisable wave-functional). The other coefficients in (8) vanish as the cut-off,  $r_0$  is taken to  $-\infty$ .

The free energy can be regulated with a Seeley-de Witt expansion of the heat-kernel

$$\text{Tr } \Gamma = \sum_{n=0}^{\infty} b_n(r_0) \left( -\frac{\partial}{\partial s} \right)^n \text{Tr } \exp \left( -s \left( \square + \hat{R}/6 \right) \right) \quad (10)$$

$$\text{Tr } \exp \left( -s \left( \square + \hat{R}/6 \right) \right) = \int d^4 x \sqrt{\hat{g}} \frac{1}{16\pi^2 s^2} \left( a_0 + s a_1(x) + s^2 a_2(x) + s^3 a_3(x) + \dots \right) \quad (11)$$

where to remove the regulator we take the proper-time separation  $s$  to zero, and  $r_0 \rightarrow -\infty$ . The only surviving contributions come from  $a_0$ ,  $a_1$  and  $a_2$ . The  $a_2$  contribution is finite and determines the Weyl anomaly [4], but the  $a_0$  and  $a_1$  contributions diverge and renormalise the boundary cosmological and Newton’s constants respectively.

Now we derived the Schrödinger equation for a scalar field, but with a little work, we can find a similar equation for all the fields of Supergravity, with the coefficients appearing in (11) being the appropriate coefficients for a conformal field of the appropriate spin. Details of this will be given in [4]. It makes sense to consider the same proper-time separation for all the fields (this is inevitable if we rewrite (11) in a superfield formalism) and so it makes sense to sum over the  $a_0$  and  $a_1$  coefficients of all the fields in the bulk spectrum, in order to determine the overall renormalisation.

The divergent coefficients of  $a_0$  and  $a_1$  are proportional to  $\sqrt{l^2 m^2 + 4} = \Delta - 2$ . In Table 1 we list the values of  $\Delta - 2$  for the fields in the bulk spectrum, originally worked out in [8]. The multiplets are labelled by an integer  $p \geq 2$  and live in representations of  $SU(4)$ . The  $a_0$  coefficients are given by  $(-1)^{2\sigma} \text{Tr} 1$  where  $\sigma$  is the spin of the field. As a result, cancellation within supermultiplets is guaranteed by the presence of equal numbers of bosonic and fermionic modes.

The  $a_1$  coefficients are given by  $\frac{1}{6} R \text{Tr} 1 - \text{Tr} E$ , where  $-\nabla^2 - E$  is the operator associated with the conformally coupled six-dimensional field (this is the operator appearing in the heat-kernel). For a conformally coupled scalar, fermion and gauge-fixed vector field this gives  $a_1 = 0$ ,  $\hat{R}/3$ , and  $-2\hat{R}/3$  respectively.

If we sum the coefficient of  $a_1$  over all the fields of the theory, the contribution to (11) can be written as

$$\int d^4x \sqrt{\hat{g}} \frac{1}{16\pi^2 s} \sum_p (\Delta - 2) a_1(x), \quad (12)$$

and denoting the values of  $a_1$  for the fields  $\phi$ ,  $\psi$ ,  $A_\mu$ ,  $A_{\mu\nu}$ ,  $\psi_\mu$ ,  $h_{\mu\nu}$  by  $s, f, v, a, r$ , and  $g$  respectively we have

$$\begin{aligned} \left( \sum (\Delta - 2) a_1 \right)_{p \geq 4} &= (-4s + 4a + r + f + 2v) \frac{p}{3} + \\ &(-105s - g - 26a - 8r - 72f - 48v) \frac{p^3}{12} + (16v + 20f + 10a + 4r + 25s + g) \frac{p^5}{12} \end{aligned} \quad (13)$$

whilst for the  $p = 3$  multiplet we have

$$\left( \sum (\Delta - 2) a_1 \right)_{p=3} = 244f + 18g + 266s + 218v + 148a + 64r. \quad (14)$$

The  $p = 2$  multiplet contains gauge fields requiring the introduction of Faddeev-Popov ghosts, whose parameters are listed in Table 2. This gives

$$\left( \sum (\Delta - 2) a_1 \right)_{p=2} = 12v - 6s + 6r + 6f + 2g + 12a \quad (15)$$

We have to deal with the sum over multiplets labelled by  $p$ . We will evaluate this divergent sum by weighting the contribution of each supermultiplet by  $z^p$ . The sum can be performed for  $|z| < 1$ , and we take the result to be a regularisation of the weighted sum for all values of  $z$ . Multiplying this by  $1/(z - 1)$  and integrating around the pole at  $z = 1$  gives a regularisation of the original divergent sum. This yields

$$\sum (\Delta - 2) a_1 = 8s + 4f + 2v \quad (16)$$

which remarkably depends only on the heat-kernel coefficients of fields in the Super-Yang-Mills theory. By decomposing a five-dimensional vector into longitudinal and transverse pieces and solving the Schrödinger equation for them, it can be seen that the heat-kernel coefficient for a vector field,  $v$ , is related to that for a four-dimensional gauge-fixed Maxwell field,  $v_0$ , as  $v = v_0 + 2s - 2s_0$  where  $s_0$  is the coefficient for a minimally coupled four-dimensional scalar (Faddeev-Popov ghost), showing  $v - 2s = v_0 - 2s_0 = g_v$  [4]. If we substitute the values for the  $a_1$  coefficients we see that the sum (16) vanishes, so that there is no overall renormalisation of the boundary Newton's constant.

As emphasised earlier other renormalisations arising from interactions of bulk Super-gravity fields should follow a similar pattern. If the boundary is taken to be Ricci-flat, we expect to observe a cancellation within supermultiplets, whereas if the boundary is non-Ricci flat we would need an additional regularisation of the sum over Kaluza-Klein modes such as we have made use of here.

## References

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Table 1: Mass spectrum. The supermultiplets (irreps of  $U(2,2/4)$ ) are labelled by the integer  $p$ . Note that the doubleton ( $p = 1$ ) does not appear in the spectrum. The  $(a, b, c)$  representation of  $SU(4)$  has dimension  $(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)/12$ , and a subscript  $c$  indicates that the representation is complex. (Spinors are four component Dirac spinors in  $AdS_5$ ).

Field	$SO(4)$ rep <sup>n</sup>	$SU(4)$ rep <sup>n</sup>	$\Delta - 2$
$\phi^{(1)}$	$(0, 0)$	$(0, p, 0)$	$p - 2, \quad p \geq 2$
$\psi^{(1)}$	$(\frac{1}{2}, 0)$	$(0, p - 1, 1)_c$	$p - 3/2, \quad p \geq 2$
$A_{\mu\nu}^{(1)}$	$(1, 0)$	$(0, p - 1, 0)_c$	$p - 1, \quad p \geq 2$
$\phi^{(2)}$	$(0, 0)$	$(0, p - 2, 2)_c$	$p - 1, \quad p \geq 2$
$\phi^{(3)}$	$(0, 0)$	$(0, p - 2, 0)_c$	$p, \quad p \geq 2$
$\psi^{(2)}$	$(\frac{1}{2}, 0)$	$(0, p - 2, 1)_c$	$p - 1/2, \quad p \geq 2$
$A_{\mu}^{(1)}$	$(\frac{1}{2}, \frac{1}{2})$	$(1, p - 2, 1)$	$p - 1, \quad p \geq 2$
$\psi_{\mu}^{(1)}$	$(1, \frac{1}{2})$	$(1, p - 2, 0)_c$	$p - 1/2, \quad p \geq 2$
$h_{\mu\nu}$	$(1, 1)$	$(0, p - 2, 0)$	$p, \quad p \geq 2$
$\psi^{(3)}$	$(\frac{1}{2}, 0)$	$(2, p - 3, 1)_c$	$p - 1/2, \quad p \geq 3$
$\psi^{(4)}$	$(\frac{1}{2}, 0)$	$(0, p - 3, 1)_c$	$p + 1/2, \quad p \geq 3$
$A_{\mu}^{(2)}$	$(\frac{1}{2}, \frac{1}{2})$	$(1, p - 3, 1)_c$	$p, \quad p \geq 3$
$A_{\mu\nu}^{(2)}$	$(1, 0)$	$(2, p - 3, 0)_c$	$p, \quad p \geq 3$
$A_{\mu\nu}^{(3)}$	$(1, 0)$	$(0, p - 3, 0)_c$	$p + 1, \quad p \geq 3$
$\psi_{\mu}^{(2)}$	$(1, \frac{1}{2})$	$(1, p - 3, 0)_c$	$p + 1/2, \quad p \geq 3$
$\phi^{(4)}$	$(0, 0)$	$(2, p - 4, 2)$	$p, \quad p \geq 4$
$\phi^{(5)}$	$(0, 0)$	$(0, p - 4, 2)_c$	$p + 1, \quad p \geq 4$
$\phi^{(6)}$	$(0, 0)$	$(0, p - 4, 0)$	$p + 2, \quad p \geq 4$
$\psi^{(5)}$	$(\frac{1}{2}, 0)$	$(2, p - 4, 1)_c$	$p + 1/2, \quad p \geq 4$
$\psi^{(6)}$	$(\frac{1}{2}, 0)$	$(0, p - 4, 1)_c$	$p + 3/2, \quad p \geq 4$
$A_{\mu}^{(3)}$	$(\frac{1}{2}, \frac{1}{2})$	$(1, p - 4, 1)$	$p + 1, \quad p \geq 4$

Table 2: Decomposition of gauge fields for the massless multiplet.

Original field	Gauge fixed fields	$\Delta - 2$	$R_{ij} = 0:$ $180a_2/R_{ijkl}R^{ijkl}$	Constant $R:$ $180a_2/R^2$
$A_\mu$ ( <b>15</b> of $SU(4)$ )	$A_i$	1	-11	29/3
	$A_0$	2	1	-1/12
	$b_{FP}, c_{FP}$	2	-1	1/12
$\psi_\mu$ ( <b>4</b> of $SU(4)$ )	$\psi_i^{\text{irr}}$	3/2	-219/2	-61/4
	$\gamma^i \psi_i$	5/2	7/2	-11/12
	$\psi_0$	5/2	7/2	-11/12
	$\lambda_{FP}, \rho_{FP}$	5/2	-7/2	11/12
	$\sigma_{GF}$	5/2	-7/2	11/12
$h_{\mu\nu}$ ( $SU(4)$ singlet)	$h_{ij}^{\text{irr}}$	2	189	727/4
	$h_{0i}$	3	-11	29/3
	$h_{00}, h_\mu^\mu$	$\sqrt{12}$	1	-1/12
	$B_0^{FP}, C_0^{FP}$	$\sqrt{12}$	-1	1/12
	$B_i^{FP}, C_i^{FP}$	3	11	-29/3